Nuclear Matter Equation of State

Akira Ohnishi (YITP, Kyoto Univ.)

- Introduction
  Why do we study EOS?

- Relativistic Mean Field description of EOS
  \( \sigma \omega \) model, Non-linear terms, Chiral RMF

- Dense Matter EOS with Hyperons
  Hyperon potentials in nuclear matter, EOS with hyperons,

- Collective flow and EOS in High-Energy Heavy-Ion Collisions
  Semi-classical transport model, Collective flows at AGS & SPS

- Summary
Introduction

Why do we study Nuclear Matter EOS?
Why do we study Nuclear Matter EOS?

Answer 1: Since bulk nuclear properties are mainly determined by nuclear matter EOS, it is important for nuclear physics.

- Nuclear Radius $\rightarrow$ Saturation of Density
  \[ R_A = r_0 A^{1/3} \quad (r_0 = 1.2 \text{ fm}) \]

- Nuclear Binding Energy (Bethe-Weizsacker Formula)
  \[
  B(A, Z) = a_{\text{vol}} A - a_{\text{surf}} A^{2/3} - a_{\text{Coulomb}} \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(N-Z)^2}{A} + a_{\text{pair}} \delta(A,Z) A^{-3/4}
  \]
Why do we study Nuclear Matter EOS?

Answer 2: Since nuclear matter EOS is decisive in compact astrophysical objects such as neutron stars, supernovae, and black hole formation, EOS is important to understand where atomic elements are made.

Why do we study Nuclear Matter EOS?

Answer 3: Since the EOS should have singularity (or at least sudden change) at phase boundary, it would be possible to catch the signal of phase transition in nuclear collisions.

Pressure and Energy Density of Free Massless Gas

\[ P = \frac{\pi^2}{90} N_B T^4 \quad \text{and} \quad \epsilon = \frac{\pi^2}{30} N_B T^4 \]

\( N_B = \text{Bosonic DOF (7/8 for Fermions)} \)

Hadron Gas \( \sim 3 \) pions (\( N_B = 3 \))

\[ P_{\pi} = \frac{\pi^2}{30} T^4 \quad \text{and} \quad \epsilon_{\pi} = \frac{\pi^2}{10} T^4 \]

QGP \( N_B = 16 \) (gluon) + 24 × 7/8 (quarks) and Bag Pressure

\[ P_{\text{QGP}} = \frac{37 \pi^2}{90} T^4 - B \quad \text{and} \quad \epsilon_{\text{QGP}} = \frac{37 \pi^2}{30} T^4 + B \]
In this lecture, I discuss several aspects of Nuclear Matter EOS

Lecture 1
(1) EOS and Mean Field in Finite Nuclei and Nuclear Matter
   → Relativistic Mean Field

Lecture 2
(2) Dense Matter EOS and Compact Astrophysical Objects
   → EOS with Hyperons, Neutron Stars, Supernovae,
      Black Hole Formation

(3) Hot and Dense Matter EOS and High-Energy Heavy-Ion Collisions
   → Nuclear Transport Model, Collective Flows
Relativistic Mean Field
Theories/Models for Nuclear Matter EOS

- **Ab initio Approach**
  - LQCD, GFMC, Variational, DBHF, G-matrix
    - Not easy to handle, Not satisfactory for phen. purposes

- **Mean Field from Effective Interactions ~ Nuclear Density Fuctionals**
  - Skyrme Hartree-Fock(-Bogoliubov)
    - Non.-Rel., Zero Range, Two-body + Three-body (or ρ-dep. two-body)
    - In HFB, Nuclear Mass is very well explained (Total B.E. ΔE ~ 0.6 MeV)
    - Causality is violated at very high densities.

- **Relativistic Mean Field**
  - Relativistic, Meson-Baryon coupling, Meson self-energies
  - Successful in describing pA scatering (Dirac Phenomenology)
Relativistic Mean Field (1)

- Relativistic Mean Field
  = Nuclear scalar and vector mean field generated by mesons
  → Why do we use relativistic framework?

- Nuclear Force is mediated by mesons
  → Let's consider meson-baryon system!
    (Entrance of Hadron Physics)

- We are also interested in Dense Matter EOS
  → Sound velocity exceeds the Speed of Light (=c) with Non.-Rel. MF

- Success of “Dirac Phenomenology”
  (Dirac Eq. for pA scattering → Spin Observables)
  → Strong Scalar and Vector Mean Fields are preferable to explain
    Spin Observables

- DBHF (Dirac-Brueckner-Hatree-Fock)
  → Successful description of nuclear matter saturation point
    based on bare NN interactions

RMF is a good starting point as a framework
of hadronic system including Nuclei and Nuclear Matter
Dirac Phenomenology

E.D. Cooper, S. Hama, B.C. Clark, R.L. Mercer, PRC47('93),297

- Dirac Eq. with Scalar + Vector pA potential
  (-400 MeV + 350 MeV)
  → Cross Section, Spin Observables
EOS in Dirac-Brueckner-Hartree-Fock

R. Brockmann, R. Machleidt, PRC42('90),1965

- Non Relativistic Brueckner Calculation
  → Nuclear Saturation Point cannot be reproduced (Coester Line)

- Relativistic Approach (DBHF)
  → Relativity gives additional repulsion, leading to successful description of the saturation point.
Relativistic Mean Field (2)

Mean Field treatment of meson field operator
= Meson field operator is replaced with its expectation value
  \( \phi(r) \rightarrow \langle \phi(r) \rangle \)

Ignoring fluctuations compared with the expectation value may be a good approximation at strong condensate.

Which Hadrons should be included in RMF?

- Baryons (1/2+)
  - p, n, Λ, Σ, Ξ, Δ, ....

- Scalar Mesons (0+)
  - σ(600), f_0(980), a_0(980), ...

- Vector Mesons (1-)
  - ω(783), ρ(770), φ(1020), ....

- Pseudo Scalar (0-)
  - π, K, η, η', ....

- Axial Vector (1+)
  - a_1, ....

We require that the meson field can have uniform expectation values in nuclear matter.

→ Scalar and Time-Component of Vector Mesons (σ, ω, ρ, ....)
σω Model (1)

Serot, Walecka, Adv. Nucl. Phys. 16 (1986), 1

- Consider only σ and ω mesons

- Lagrangian

\[
L = \bar{\psi} \left( i \gamma^\mu \partial_\mu - M + g_s \sigma - g_v \omega \right) \psi \\
+ \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m_s^2 \sigma^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} m_v^2 \omega_\mu \omega^\mu \\
( F_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu )
\]

- Equation of Motion

  - Euler-Lagrange Equation

\[
\frac{\partial}{\partial x^\mu} \left[ \frac{\partial L}{\partial (\partial_\mu \phi_i)} \right] - \frac{\partial L}{\partial \phi_i} = 0
\]

\[
\sigma : \left[ \partial_\mu \partial^\mu + m_s^2 \right] \sigma = g_s \bar{\psi} \psi
\]

\[
\omega : \partial_\mu F^{\mu \nu} + m_v^2 \omega^\nu = g_v \bar{\psi} \gamma^\nu \psi \quad \rightarrow \quad \left[ \partial_\mu \partial^\mu + m_v^2 \right] \omega^\nu = g_v \bar{\psi} \gamma^\nu \psi
\]

\[
\psi : \left[ \gamma^\mu \left( i \partial_\mu - g_v V_\mu \right) - \left( M - g_s \sigma \right) \right] \psi = 0
\]
EOM of $\omega$ (for beginners)

- **Euler-Lagrange Eq.**

\[ \partial_\mu F^{\mu\nu} + m_\nu \omega^\nu = g_\nu \bar{\psi} \gamma^\nu \psi \]

- **Divergence of LHS and RHS**

\[ \partial_\nu \partial_\mu F^{\mu\nu} + m_\nu^2 (\partial_\nu \omega^\nu) = m_\nu^2 (\partial_\nu \omega^\nu) = g_\nu (\partial_\nu \bar{\psi} \gamma^\nu \psi) = 0 \]

LHS: derivatives are sym. and $F_{\mu\nu}$ is anti-sym.
RHS: Baryon Current = Conserved Current

- **Put it in the Euler-Lagrange Eq.**

\[ \partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu \omega^\nu - \partial^\nu \omega^\mu) = \partial_\mu \partial^\mu \omega^\nu - \partial^\nu (\partial_\mu \omega^\mu) = \partial_\mu \partial_\mu \omega^\nu \]
**Schroedinger Eq. for Upper Component**

- **Dirac Equation for Nucleons**
  \[
  \left( i \gamma \partial - \gamma^0 U_v - M - U_s \right) \psi = 0 \ , \quad U_v = g_\omega \omega \ , \quad U_s = - g_\sigma \sigma
  \]

- **Decompose 4 spinor into Upper and Lower Components**
  \[
  \begin{pmatrix}
  E - U_v - M - U_s & i \sigma \cdot \nabla \\
  -i \sigma \cdot \nabla & -E + U_v - M - U_s
  \end{pmatrix}
  \begin{pmatrix}
  f \\
  g
  \end{pmatrix} = 0
  \]
  \[
  g = -\frac{i}{E + M + U_s - U_v} (\sigma \cdot \nabla) f
  \]
  \[
  (E - M - U_v - U_s) f = -i (\sigma \cdot \nabla) g
  \]

- **Erase Lower Component (assuming spherical sym.)**
  \[
  -i (\sigma \cdot \nabla) g = - (\sigma \cdot \nabla) \frac{1}{X} (\sigma \cdot \nabla) f = -\frac{1}{X} \nabla^2 f - \frac{1}{r} \left[ \frac{d}{dr} \frac{1}{X} \right] (\sigma \cdot r) (\sigma \cdot \nabla) f = -\nabla \frac{1}{X} \nabla f + \frac{1}{r} \left[ \frac{d}{dr} \frac{1}{X} \right] (\sigma \cdot l) f
  \]
  \[
  (\sigma \cdot r) (\sigma \cdot \nabla) = (r \cdot \nabla) + i \sigma \cdot (r \times \nabla) = r \cdot \nabla - \sigma \cdot l
  \]

- **“Schroedinger-like” Eq. for Upper Component**
  \[
  -\nabla \frac{1}{E + M + U_s - U_v} \nabla f + (U_s + U_v + U_{LS} (\sigma \cdot l)) f = (E - M) f
  \]
  \[
  U_{LS} = \frac{1}{r} \left[ \frac{d}{dr} \frac{1}{E + M + U_s - U_v} \right] < 0 \quad \text{on surface}
  \]
  \[
  (U_s, U_v) \sim (-350 \text{ MeV}, 280 \text{ MeV}) \rightarrow \text{Small Central}(U_s + U_v), \text{ Large LS } (U_s - U_v)
  \]
Various Ways to Evaluate Non.-Rel. Potential

From Single Particle Energy

\[
\left(\gamma^0 (E - U_v) + i \gamma \cdot \nabla - (M + U_s) \right) \psi = 0 \rightarrow (E - U_v)^2 = p^2 + (M + U_s)^2
\]

\[
E = \sqrt{p^2 + (M + U_s)^2} + U_v \approx E_p + \frac{M}{E_p} U_s + U_v + \frac{p^2}{2 E_p^3} U_s^2
\]

\[
(E_p = \sqrt{p^2 + M^2})
\]

Schroedinger Equivalent Potential (Uniform matter)

\[
- \frac{\nabla^2}{2M} f + \left[ U_s + \frac{E}{M} U_v + \frac{U_s^2 - U_v^2}{2M} \right] f = \frac{E + M}{2M} (E - M) f
\]

\[
U_{\text{SEP}} \approx U_s + \frac{E}{M} U_v
\]

Anyway, slow baryons feel Non.-Rel. Potential,

\[
U \approx U_s + U_v = - g_s \sigma + g_v \omega
\]
Nuclear Matter in $\sigma\omega$ Model

Serot, Walecka, Adv.Nucl.Phys.16 (1986),1

Uniform Nuclear Matter

$$E/V = \gamma_N \int^{P_F} \frac{d^3 p}{(2\pi)^2} E^* + \frac{1}{2} m_s^2 \sigma^2 - \frac{1}{2} m_v^2 \omega^2 + g_v \rho_B \omega$$

$$\sigma = \frac{g_s}{m_s} \rho_s = \frac{g_s}{m_s} \int^{P_F} \gamma_N \frac{d^3 p}{(2\pi)^2} M^*$$

$$\omega = \frac{g_v}{m_v^2} \rho_B = \gamma_N \frac{g_v}{m_v^2} \int^{P_F} \frac{d^3 p}{(2\pi)^3}$$

$\gamma_N = \text{Nucleon degeneracy}$

($=4 \text{ in sym. nuclear matter}$)

Problem: EOS is too stiff

$K \sim (500-600) \text{ MeV}!$

$\rightarrow \text{How can we solve ?}$
RMF with Non-Linear Meson Int. Terms

Too stiff EOS in the simplest RMF (σω model) is improved by introducing non-linear terms (σ⁴, ω⁴)

- Fit B.E. of Stable as well as Unstable (n-rich) Nuclei
- Three Mesons (σ,ω,ρ) are included

Meson Self-Energy Term (σ,ω)

\[
\mathcal{L} = \overline{\psi}_N (i\not{\partial} - M - g_\sigma \sigma - g_\omega \not{\partial} - g_\rho \tau^a \rho^a) \psi_N
\]
\[
+ \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 \left[ \frac{1}{3} g_2 \sigma^3 - \frac{1}{4} g_3 \sigma^4 \right]
\]
\[
- \frac{1}{4} W_{\mu\nu} W_{\mu\nu} + \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu - \frac{1}{4} R^{a\mu\nu} R^a_{\mu\nu} + \frac{1}{2} m_\rho^2 \rho^{a\mu} \rho^a_\mu + \frac{1}{4} c_3 (\omega_\mu \omega^\mu)^2
\]
\[
+ \overline{\psi}_e (i\not{\partial} - m_e) \psi_e + \overline{\psi}_\nu i\not{\partial} \psi_\nu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]

\[
W_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu,
\]
\[
R^a_{\mu\nu} = \partial_\mu \rho^a_\nu - \partial_\nu \rho^a_\mu + g_\rho \epsilon^{abc} \rho^b_\mu \rho^{c\nu},
\]
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]
RMF with Non-Linear Meson Int. Terms

Are the Lagrangian parameters are well determined?

\[ \mathcal{L} = \mathcal{L}_{\text{free}}(\psi, \sigma, \omega, \rho, \ldots) + \bar{\psi} \left[ g_\sigma \sigma - g_\omega \gamma^0 \omega - g_\rho \tau_3 \gamma^0 \rho \right] \psi \\
+ c_\omega \omega^4/4 - V_\sigma(\sigma), \quad (3) \]

\[ V_\sigma = \begin{cases} 
\frac{1}{3} g_3 \sigma^3 + \frac{1}{4} g_4 \sigma^4 & \text{(NL1, NL3, TM1)} \\
-a_\sigma f_{\text{SCL}}(\sigma/f_\pi) & \text{(SCL)} 
\end{cases}, \quad (4) \]

- Linear terms, Meson-Nucleon Coupling → Well determined
- Meson interaction terms → Different in RMF parameterization
- Negative Coef. of \( \sigma^4 < 0 \) → Vacuum is unstable

Is there any way to “Derive” RMF Lagrangian?

→ Symmetry in QCD

---

AO, Jido, Sekihara, Tsubakihara, in prep.

Ohnishi, YITP Colloq., 2008/05/28
Chiral RMF
Nuclear Many-Body Theory preserving Chiral Sym.?

- **Chiral Symmetry**
  \[ \psi_R = \frac{1}{2} (1 + \gamma_5) \psi \rightarrow U_R \psi_R, \quad \psi_L = \frac{1}{2} (1 - \gamma_5) \psi \rightarrow U_L \psi_L \]

- **Symmetry in QCD with small quark mass**
  Kinetic term = invariant: \[ \bar{\psi} i \gamma^\mu D_\mu \psi = \bar{\psi}_R i \gamma^\mu D_\mu \psi_R + \bar{\psi}_L i \gamma^\mu D_\mu \psi_L \]
  mass term ≠ invariant: \[ \bar{\psi} \psi = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \]

- **Should be kept in Nuclear Lagrangian**
  Problem: Nucleon cannot have mass!
  Solution: Spontaneous breaking of Chiral Sym.

\[ L_{\sigma M} = \frac{1}{2} \left( \partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \partial^\mu \pi \right) \]
\[ - \frac{\lambda}{4} \left( \sigma^2 + \pi^2 \right)^2 + \frac{\mu^2}{2} \left( \sigma^2 + \pi^2 \right) + c \sigma \]
\[ + \bar{N} i \partial_\mu \gamma^\mu N - g_\sigma \bar{N} \left( \sigma + i \pi \tau \gamma_5 \right) N \]

- *Gell-Mann, Levy, 1960; Nambu, Jona-Lasino, 1961*

Ohnishi, YITP Colloq.,
Chiral Collapse Problem (Lee-Wick Vacuum)

At finite $\rho_B$, Nucleon Fermi Integral favors smaller $\sigma$ → Chiral Sym. is restored below $\rho_0$ (Chiral Collapse) *Lee, Wick, 1974*

**Prescriptions**

- $\sigma\omega$ coupling (too stiff EOS) *Boguta 1983, Ogawa et al. 2004*

- Loop effects (unstable at large $\sigma$) *(Matsui-Serot, 1982, Glendenning 1988, Prakash-Ainsworth 1987, Tamenaga et al. 2006)*

- Higher order terms (unstable at large $\sigma$) *(Hatsuda-Prakash 1989, Sahu-Ohnishi 2000)*

- **Dielectric (Glueball) Field representing scale anomaly** *(Furnstahl-Serot 1993, Heide-Rudaz-Ellis 1994, Papazoglou et al. (SU(3)) 1998)*


*but .....*
Many of the proposed prescriptions fail.

- Naïve $\phi^4$ (Unstable)
- Lee-Wick Fermion Loop (Unstable)
- SCL-LQCD

$\sigma\omega$ coupling (too stiff EOS)

Higher order terms (Unstable)

Phenom.
Chiral RMF based on SCL-LQCD

- Relativistic Mean Field model
  - Effective Lagrangian consisting Baryons and Mesons
  - Attractive Scalar Field ($\sigma$) + Repul. Vector Field ($\omega$) → Matter Saturation

\[ \mathcal{L}_\chi = \bar{\psi}_N \left[ i \gamma^\mu \left( \partial_\mu - g_\sigma \frac{\sigma}{2} \right) + g_\omega \frac{\sigma}{2} \gamma^\mu \gamma^5 + \frac{1}{2} \left( \partial^\mu \sigma \partial_\mu \sigma + \partial^\mu \pi \cdot \partial_\mu \pi \right) - V_\sigma(\sigma, \pi) \right. \]

\[ \left. - \frac{1}{4} W^{\mu\nu} W_{\mu\nu} + \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu + \frac{c_0}{4} (\omega^\mu \omega_\mu)^2 - \frac{1}{4} R^{\mu\nu} \cdot R_{\mu\nu} + \frac{1}{2} m_\rho^2 \rho^\mu \cdot \rho_\mu \right] \]

- Linear $\sigma$ Model
  \[ V_\sigma = -\frac{1}{2} \mu \sigma^2 + \frac{1}{4} \lambda \sigma^4 \]

- SCL-LQCD (Zero T)
  \[ V_\sigma = \frac{1}{2} a_\sigma \sigma^2 - b_\sigma \log \sigma \]

→ Agree with Phen. RMF results at large chiral condensates.
Chiral RMF based on SCL-LQCD

Nuclear Matter EOS

- Gives “Medium” EOS (K ~ 280 MeV), Comparable to Phen. RMF

Bulk properties of nuclei

- B.E./Nucleon, Charge radii → Comparable to High Quality Phen. RMF

This would be the first step of giving Nuclear Density Functional from QCD
Binding Energies in Chiral and Non-Chiral RMF

Non-Chiral High Precision RMF: TM1 & 2, NL1, NL3
(Sugahara, Toki, 1994; Reinhard et al., 1986; Lalazissis, Koenig, Ring, 1997)

Log term from Scale Anomaly: I/110, IF/110, VIIIIF/100
Chiral Symmetric, No Instability, with Glueball  \[ V_\sigma = -\lambda^4 \log \sigma^2 \]
(Heide, Rudas, Ellis, 1994)

Quark Meson Coupling model

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>exp.</th>
<th>SCL</th>
<th>TM1</th>
<th>TM2</th>
<th>NL1</th>
<th>NL3</th>
<th>I/110</th>
<th>IF/110</th>
<th>VIIIIF/100</th>
<th>QMC-I</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{12}\text{C})</td>
<td>7.68</td>
<td>7.09</td>
<td>-</td>
<td>7.68</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(^{16}\text{O})</td>
<td>7.98</td>
<td>8.06</td>
<td>-</td>
<td>7.92</td>
<td>7.95</td>
<td>8.05</td>
<td>7.35</td>
<td>7.86</td>
<td>7.18</td>
<td>5.84</td>
</tr>
<tr>
<td>(^{28}\text{Si})</td>
<td>8.45</td>
<td>8.02</td>
<td>-</td>
<td>8.47</td>
<td>8.25</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(^{40}\text{Ca})</td>
<td>8.55</td>
<td>8.57</td>
<td>8.62</td>
<td>8.48</td>
<td>8.56</td>
<td>8.55</td>
<td>7.96</td>
<td>8.35</td>
<td>7.91</td>
<td>7.36</td>
</tr>
<tr>
<td>(^{48}\text{Ca})</td>
<td>8.67</td>
<td>8.62</td>
<td>8.65</td>
<td>8.70</td>
<td>8.60</td>
<td>8.65</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(^{58}\text{Ni})</td>
<td>8.73</td>
<td>8.54</td>
<td>8.64</td>
<td>-</td>
<td>8.70</td>
<td>8.68</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(^{90}\text{Zr})</td>
<td>8.71</td>
<td>8.69</td>
<td>8.71</td>
<td>-</td>
<td>8.71</td>
<td>8.70</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(^{116}\text{Sn})</td>
<td>8.52</td>
<td>8.51</td>
<td>8.53</td>
<td>-</td>
<td>8.52</td>
<td>8.51</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(^{198}\text{Pb})</td>
<td>7.87</td>
<td>7.87</td>
<td>7.87</td>
<td>-</td>
<td>7.89</td>
<td>-</td>
<td>7.33</td>
<td>7.54</td>
<td>7.44</td>
<td>7.25</td>
</tr>
<tr>
<td>(^{208}\text{Pb})</td>
<td>7.87</td>
<td>7.87</td>
<td>7.87</td>
<td>-</td>
<td>7.89</td>
<td>7.88</td>
<td>7.33</td>
<td>7.54</td>
<td>7.44</td>
<td>7.25</td>
</tr>
</tbody>
</table>
Extention to Flavor SU(3)
→ Chiral Potential from SCL-LQCD
  + Determinant Int. ($U_A(1)$ anomaly)
  + Explicit breaking term

$$U_{\sigma \zeta} = -a \log(\det MM^\dagger) + b \text{tr}(MM^\dagger)$$
$$+ c_\sigma \sigma + c_\zeta \zeta + d (\det M + \det M^\dagger).$$

Normal, Single & Double $\Lambda$, $\Sigma$ atom, EOS ($\sim$ FP), ....
Summary of Lecture 1

- Nuclear Matter EOS is important in various aspects of Nuclear Physics
- Relativistic Mean Field may be a good starting point to describe hadronic (baryon and meson) systems.
  - Relativistic $\rightarrow$ Saturation, Causality
  - Based on successes of Dirac Phenomenology and DBHF
  - Covariant Density Functional
    - It is desirable to obtain $E/V$ (energy density) in fundamental theories. (Renormalizability is not required.)
  - We can re-write RMF equations in Schroedinger-like eqs. We should consider it as a method to parameterize DF in a transparent manner.
  - Higher order terms / Density dependence of the coupling constants (not mentioned) $\rightarrow$ Necessary for precise description of nuclei, but need foundations of extension.
Relativistic EOS
of Supernova Matter with Hyperons
Supernova Explosion from Nucl. Phys. Point of View

Supernovae DO NOT EXPLODE in theor. calculation at present with realistic microphysics inputs. → How can we succeed?

Multi-Dim. Hydro (Instability)+Additional Energy Release (10 %-factor 10)

EOS tables
Lattimer-Swesty (1981)
Rel. (Shen) EOS (1998)
→ How to extend?

Realistic ν-A int.
Nuclear Dist.
Exact ν transfer
.....
Hyperons in Dense Matter

What appears at high density?

- Nucleon superfluid \( ^3S_1, ^3P_2 \)
- Pion condensation, Kaon condensation, Baryon Rich QGP, Color SuperConductor (CSC), Quarkyonic Matter, ....

Hyperons

Tsuruta, Cameron (66); Langer, Rosen (70); Pandharipande (71); Itoh(75); Glendenning; Weber, Weigel; Sugahara, Toki; Schaffner, Mishustin; Balberg, Gal; Baldo et al.; Vidana et al.; Nishizaki, Yamamoto, Takatsuka; Kohno, Fujiwara et al.; Sahu, Ohnishi; Ishizuka, Ohnishi, Sumiyoshi, Yamada; ...

Nobody says “Hyperons do not appear in neutron star core”!

\[ Y \text{ appears when } \mu_B = E_F(n) + U(n) \geq M(Y) + U(Y) + Q_Y \mu_e \]
Hyperons in Supernova Matter

- Problems to include hyperons in Supernova Matter EOS
  - Uncertainties of hyperon potentials $U_\gamma(\rho)$ → \textit{Recent Hypernuclear Phys.} (e.g. Balberg, Gal, 1997)
  - Density may not be very high in supernova → \textit{Needed in cooling stage}

- Attractive $U_\Sigma$
- Repulsive $U_\Sigma$

We include recent hyperon info. in supernova matter EOS

\[ U_\Lambda, U_\Sigma, U_\Xi = \text{-30 MeV} \]

\[ U_\Lambda = \text{-30 MeV} \]
\[ U_\Sigma = \text{-30 MeV} \]
\[ U_\Xi = \text{-30 MeV} \]

Sahu, AO, 2003
$\Sigma$ Potential in Nuclear Matter

- $U_\Lambda(\rho_0) \sim -30$ MeV: Well known from single particle energies

- Naïve expectation
  = Quark Number (ud number) Scaling
  $U_\Lambda \sim 2/3 \ U_N \rightarrow U_\Sigma \sim 2/3 \ U_N \sim -30$ MeV

- Problems with $\Sigma$
  - Only one bound state $^4_\Sigma$He (Too light!)
  - Continuum (Quasi-Free) Spectroscopy is necessary

$\pi$ $K$ $N$ $Y$ $d$ $p$ $s$

$30$ MeV

---

Tsubakihara, Maekawa, AO, EPJA33('07),295.

$\omega = \frac{q^2}{2M_Y^*} + \Delta M + U(Y) - U(N)$
Σ Potential in Nuclear Matter

Cont. Spec. Theory = Distorted Wave Impulse Approx. (DWIA)

\[
\frac{d^2 \sigma}{dE_K d \Omega_K} = \beta \left[ \frac{d \sigma}{d \Omega} \right]_{\text{Element}}^{\text{Element}} S(E, q)
\]

Kinematical Factor

Elem. Cross Sec.

Strength Func.

- Large \((\omega, q)\) range \(\rightarrow\) Important to respect On-Shell Kinematics
- Kinematics depends on Reaction Point with Hyperon Potential

Harada, Hirabayashi, NPA744('04), 323.
Kohno, Fujiwara, Kawai, et al.
PTP112('04)895

\[ K \rightarrow K \Lambda \]

\[ N \rightarrow N \Sigma \]
\[\Sigma Potential in Nuclear Matter\]

Maekawa, Tsubakihara, Matsumiya, AO, in preparation.

- DWIA with Local Optimal Fermi Averaging t-matrix (DWIA-LOFAt)
- Green's Func. Method + Reaction Point Deps. of t-matrix

\[
\frac{d^2\sigma}{dE_Kd\Omega_K} = \frac{p_K E_K}{(2\pi)^2 \nu_{inc}} R_Y(E_Y) \quad R_Y(E_Y) = -\frac{1}{\pi} \text{Im} \langle \bar{t}(r ') \rangle \left\langle \frac{1}{E_Y - H_Y + i\varepsilon} \bar{t}(r) \right\rangle
\]

\[\bar{t}(r,\omega,q) = \int dN \frac{t(s,t)p_N(p_N)\delta^{(4)}(p_1(r)+p_2(r)-p_3(r)-p(r))}{\int dN \rho(p_N)\delta^{(4)}(p_1(r)+p_2(r)-p_3(r)-p(r))} E_i = \sqrt{p_i^2 + m_i^2(r)^2} \approx m_i + \frac{p_i^2}{2m_i} + V_i, \quad m_i(r)^2 = m_i^2 + 2m_i V_i(r)\]

- After careful treatment of K+ potential, Elementary cross section, Angular distribution, ....
  we analyze the recently measured $\Sigma^-$ production spectrum

(Saha, Noumi et al. (KEK-E438), PRC70('04)044613)
$\Sigma$ Potential in Nuclear Matter

Maekawa, Tsubakihara, Matsumiya, AO, in preparation.

$\frac{d^2 \sigma}{d E_K d \Omega_K}$

$^{28}\text{Si}(\pi^-, K^+)_{p=1.20 \text{ GeV/c}, 6 \text{ deg.}}$

$U_{\Sigma}(\rho_0) \sim +15 \text{ MeV} - i \text{ 10 MeV}$

with Woods-Saxon potential, no Atomic shift fit
**Ξ Potential in Nuclear Matter**

- Currently accepted value: \( U_{\Xi} \sim -14 \text{ MeV} \)

Twin hypernuclear form., Spectrum shape in the bound state region  
(Aoki et al. PLB355('95),45; Fukuda et al. PRC58('98),1306; Khaustov et al. PRC61('00), 054603)

- Absolute values of \(^{12}\text{C}(K^-,K^+)\) spectra → Still Difficult to Understand

- Large q → Spectrum may depend on detailed nuclear structure


Matsumiya, et al.  
(Coupled Channel AMD)

Let's wait for J-PARC results

Ohnishi, CNS Summer School, 2008/08/26-09/01
“Stars” of Hyperon Potentials (A la Michelin)

- $U_\Lambda(\rho_0) \sim -30$ MeV

- Bound State Spectroscopy + Continuum Spectroscopy

- $U_\Sigma(\rho_0) > +15$ MeV

- Continuum (Quasi-Free) spectroscopy with Local Optimal Fermi Averaging t-matrix (LOFAt)

- Atomic shift data (attractive at surface) should be respected.

- $U_\Xi(\rho_0) \sim -14$ MeV

- No confirmed bound state, No atomic data, High mom. transf., .... → Small Potential Deps.

- Continuum low-res. spectrum shape → – 14 MeV

- Spin-Isospin deps. (π exch.) → Deformation → Spectrum shape may be modified.
Relativistic EOS of Supernova Matter with Hyperons

- Extention of the Relativistic (Shen) EOS to SU$_f$(3) with updated Hyperon Potentials in Nuclear Matter

- Relativistic (Shen) EOS (Shen, Toki, Oyamatsu, Sumiyoshi, PTP 100('98), 1013)
  Rel. Mean Field (RMF) + Local Density Approx. (Nuclear Formation)

- SU$_f$(3) Extension of RMF (Schaffner, Mishustin, PRC53 (1996), 1416)
  Coupling ~ Quark Number Counting

<table>
<thead>
<tr>
<th>$g_{MB}$</th>
<th>$\sigma$</th>
<th>$\zeta$</th>
<th>$\omega$</th>
<th>$\rho$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>10.0289</td>
<td>0</td>
<td>12.6139</td>
<td>4.6783</td>
<td>0</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>6.21</td>
<td>6.67</td>
<td>8.41</td>
<td>0</td>
<td>-5.95</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>4.36 (6.21)</td>
<td>6.67</td>
<td>8.41</td>
<td>$2g_{\rho N}$</td>
<td>-5.95</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>3.11 (3.49)</td>
<td>12.35</td>
<td>4.20</td>
<td>4.63</td>
<td>-11.89</td>
</tr>
</tbody>
</table>

- $g_{\sigma Y}$ is tuned to fit Hyperon Potential in Nuclear Matter
  $U_{\Lambda} = -30$ MeV, $U_{\Sigma} = +30$ MeV, $U_{\Xi} = -15$ MeV

- Nuclear Formation is included using Shen EOS table
**Tolman-Oppenheimer-Volkoff (TOV) equation**

**TOV Eq. = General Relativistic Balance of pressure and gravity**

\[
\frac{dP}{dr} = -G \frac{(\varepsilon/c^2 + P/c^2)(M + 4\pi r^3 P/c^2)}{r^2(1 - 2GM/rc^2)}
\]

\[
\frac{dM}{dr} = 4\pi r^2 \varepsilon/c^2 , \quad \frac{dP}{dr} = \frac{dP}{d\varepsilon} \frac{d\varepsilon}{dr}
\]

\[
P = P(\varepsilon) , \quad \frac{dP}{d\varepsilon} = \frac{dP}{d\varepsilon}(\varepsilon) \quad (EOS)
\]

Neutron Star Mass = M(R) where P(R)=0

When you make a new EOS, please check the NS mass!
Neutron Star


- Hyperon Effect is DRASTIC
  - $M_{\text{max}}=2.1 \text{ Msun} \rightarrow 1.56 \text{ Msun}$
  - Composition $Y_\Lambda \sim Y_n$
  - Large fraction of $\Xi$

- Thermal (free) pions can admix at $\rho > 1.5 \rho_0$


Shen

Schaffner-Mishustin

New

Central Density (g/cc)

Neutron Star Matter

$\text{Shen (Ne}_\mu)$
$\text{NY}_\mu$
$\text{NY}\pi\mu$
$\text{NY(Att.)}_\mu$

$\text{NY}_\mu$
$\text{NY}\pi\mu$
$\text{NY(Att.)}_\mu$

$\Lambda$

$\Xi$

$\pi$

$\Sigma^0+\Xi^0$

$\Sigma^+$

$\Sigma^-$

$\Sigma^0$

$\Sigma^+$

$\Sigma^-$

$\Sigma^0$

$\Sigma^+$

$\Sigma^-$

$\Sigma^0$
Finite Temperature and Supernova

Example: T=10 MeV, Ye = 0.4

Λ starts to increase at ρ ~ 2 ρ₀, becomes significant at ρ ~ 3ρ₀.

T=10 MeV, Y_C=0.4

Prompt explosion
(without ν transport)
→ Almost no change
(Expl. E. increase ~ (0.1-0.5 %))

Low density and High Ye suppresses Hyperons in the Early Stage
Where Do We See Hyperons?

Hyperon Fraction is sensitive to Ye, T, and \( \rho_B \).

- Ye \sim 0 \text{ (Neutron Star)} \rightarrow \rho_B > 2 \rho_0
- Ye \sim 0.4 \text{ (Supernova, early stage)} \rightarrow T > 40 \text{ MeV or } \rho_B > 3 \rho_0

Hyperons would be important in Late Stages
Proto neutron star cooling, Black Hole Formation

NYe, Y_C=0.4

T (MeV)

0 10 20 30 40 50 60 70 80 90 100

0 0.1 % 1 % 2 5

S/B=10

Y/B=10 %

\( \rho_B (\text{fm}^{-3}) \)

Prompt Expl. (15 Msun)
Hyperons during Black Hole Formation

- Hyperons appear abundantly during Black Hole Formation Processes
  - Off-Center: Large $T \rightarrow \Sigma > \Xi$
  - Center: Large $\rho_B \rightarrow \Sigma < \Xi$

Summary (1) of Lecture 2

Hyperons are included in the Relativistic (Shen) EOS with recently accepted Hyperon Potentials in Nuclear Matter, $U_\Lambda = -30$ MeV, $U_\Sigma = +30$ MeV, $U_\Xi = -15$ MeV

http://nucl.sci.hokudai.ac.jp/~chikako/EOS

$\rho = 10^{5.1-15.4}$ g/cc, $T=0-100$ MeV, $Y_e=0-0.56$


Hyperon effects:
- Decisive in Nstar
- Small in SNe (early)
- Significant in BH formation.

Japan Proton Accelerator Research Complex (J-PARC) data will come soon.
Stay Tuned!
Nuclear Transport Models for Heavy-Ion Collisions and Collective Flows
Study of Hot and Dense Hadronic Matter
→ Particle Yield, Collective Dynamics (Flow), EOS, .....

JAMming on the Web, linked from http://www.jcprg.org/
Nuclear Mean Field

MF has on both of $\rho$ and $p$-deps.

- $\rho$ dep.: $(\rho_0, E/A) = (0.15 \text{ fm}^{-3}, -16.3 \text{ MeV})$ is known
  Stiffness is not known well

- $p$ dep.: Global potential up to $E=1$ GeV is known from $pA$ scattering
  \[ U(\rho_0, E) = U(\rho_0, E=0) + 0.3E \]

Ab initio Approach; LQCD, GFMC, DBHF, G-matrix, ....
→ Not easy to handle, Not satisfactory for phen. purposes

Effective Interactions (or Energy Functionals):
Skyrme HF, RMF, ...

\[ U(E) = U(0) + 0.3E \]
**HIC Transport Models: Major Four Origins**

- **Nuclear Mean Field Dynamics**
  - Basic Element of Low Energy Nuclear Physics, and Critically Determines High Density EOS / Collective Flows
  - TDHF → Vlasov → BUU

- **NN two-body (residual) interaction**
  - Main Source of Particle Production
  - Intranuclear Cascade Models

- **Partonic Interaction and String Decay**
  - Main Source of high pT Particles at Collider Energies
  - JETSET + (previous) PYTHIA (Lund model) → (new) PYTHIA

- **Relativistic Hydrodynamics**
  - Most Successful Picture at RHIC
HIC Models: History

1970's~
- TDHF

1980's~
- +Gauss
- +Collision
- Vlasov
- Classical Coll.
- Anti Sym.
- BUU
- RBUU
- RMF

1990's~
- AMD
- AMD-V
- AMD-QL
- BEM
- UrQMD
- HSD
- JAM
- JAM-RQMD/S

2000's~
- +Geometry
- 3D Hydro
- Hydro+Jet
- JFS

- Jet+String
- +MF
- +Fluc.
- HIJING
- RQMD
- QMD
- Hydro
- AMPT
- Hydro+JAM
**TDHF and Vlasov Equation**

- **Time-Dependent Mean Field Theory (e.g., TDHF)**

\[ i \hbar \frac{\partial \phi_i}{\partial t} = \hbar \phi_i \]

- **Density Matrix**

\[ \rho(r, r') = \sum_{\text{occ}} \phi_i(r) \phi_i^*(r') \rightarrow \rho_W = f \text{ (phase space density)} \]

- **TDHF for Density Matrix**

\[ i \hbar \frac{\partial \rho}{\partial t} = [\hbar, \rho] \rightarrow \frac{\partial f}{\partial t} = \{\hbar_W, f\}_{P.B.} + O(\hbar^2) \]

- **Wigner Transformation and Wigner-Kirkwood Expansion**

(Ref.: Ring-Schuck)

\[ O_W(r, p) \equiv \int d^3s \exp(-i p \cdot s / \hbar) <r + s/2|O|r - s/2> \]

\[(AB)_W = A_W \exp(i \hbar \Lambda) B_W \quad \Lambda \equiv \nabla' \cdot \nabla_p - \nabla' \cdot \nabla_r \quad (\nabla' \text{ acts on the left})\]

\[ [A, B]_W = 2i A_W \sin(\hbar \Lambda / 2) B_W = i \hbar \{A_W, B_W\}_{P.B.} + O(\hbar^3) \]
Test Particle Method

- Vlasov Equation
  \[
  \frac{\partial f}{\partial t} - \{ h_w, f \}_{P.B.} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f - \nabla U \cdot \nabla_p f = 0
  \]

- Classical Hamiltonian
  \[
  h_w(r, p) = \frac{p^2}{2m} + U(r, p)
  \]

- Test Particle Method (C. Y. Wong, 1982)
  \[
  f(r, p) = \frac{1}{N_0} \sum_{i}^{A N_0} \delta(r-r_i) \delta(p-p_i) \quad \rightarrow \quad \frac{d r_i}{d t} = \nabla_p h_w, \quad \frac{d p_i}{d t} = -\nabla_r h_w,
  \]

Mean Field Evolution can be simulated by Classical Test Particles
→ Opened a possibility to Simulate High Energy HIC including Two-Body Collisions in Cascade
**BUU (Boltzmann-Uehling-Uhlenbeck) Equation**

- **BUU Equation** (Bertsch and Das Gupta, Phys. Rept. 160(88), 190)

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f - \nabla U \cdot \nabla_p f = I_{\text{coll}}[f]
\]

\[
I_{\text{coll}}[f] = -\frac{1}{2} \int \frac{d^3 p_2}{(2\pi \hbar)^3} \frac{d \sigma}{d \Omega} \nu_{12} \times \left[ f f_2 (1-f_3)(1-f_4) - f_3 f_4 (1-f)(1-f_2) \right]
\]

- **Incorporated Physics in BUU**
  - Mean Field Evolution
  - (Incoherent) Two-Body Collisions
  - Pauli Blocking in Two-Body Collisions

- **One-Body Observables (Particle Spectra, Collective Flow, ..)**

- **Event-by-Event Fluctuation (Fragment, Intermittency, ..)**
Comparison of TDHF, Vlasov and BUU(VUU)

- Ca+Ca, 40 A MeV
Relativistic Mean Field (II)

- Dirac Equation
  \[ \left( i \gamma \partial - \gamma^0 (U_v - M - U_s) \right) \psi = 0 \], \( U_v = g_\omega \omega \), \( U_s = -g_\sigma \sigma \)

- Schroedinger Equivalent Potential
  \[ \begin{pmatrix} E - U_v - M - U_s & i \sigma \cdot \nabla \\ -i \sigma \cdot \nabla & -E + U_v - M - U_s \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0 \]

\[ U_{\text{sep}} \sim U_s + \frac{E}{m} U_v = -g_\sigma \sigma + \frac{E}{m} g_\omega \omega \]

\[ = -g_\sigma^2 \rho_s + \frac{E}{m} \frac{g_\omega^2}{m_\omega^2} \rho_B \]

Saturation: -Scalar+Baryon Density
Linear Energy Dependence: Good at Low Energies, Bad at High Energies (We need cut off!)

(Sahu, Cassing, Mosel, AO, Nucl. Phys. A672 (2000), 376.)
Phenomenological Mean Field

- Skyrme type $\rho$-Dep. + Lorentzian $p$-Dep. Potential

$$V = \sum_i V_i = \int d^3 r \left[ \frac{\alpha}{2} \left( \frac{\rho}{\rho_0} \right)^2 + \frac{\beta}{\gamma+1} \left( \frac{\rho}{\rho_0} \right)^{\gamma+1} \right]$$

$$+ \sum_k \int d^3 r \int d^3 p \int d^3 p' \frac{C_{ex}^{(k)}}{2\rho_0} \frac{f(r,p)f(r,p')}{1+(p-p')^2/\mu_k^2}$$

Isse, AO, Otuka, Sahu, Nara, Phys.Rev. C 72 (2005), 064908
Collective Flow and EOS: Old Problem?

  - Hydrodynamics suggested the Existence of Flow.
  - Strong Collective Flow suggests Hard EOS
- 1980's-1990's: Deeper Discussions in Wider $E_{\text{inc}}$ Range
  - Momentum Dep. Pot. can generate Strong Flows.
  - $E_{\text{inc}}$ depts. implies the importance of Momentum Deps.
  - Flow Measurement up to AGS Energies.
- 2000's: Extension to SPS and RHIC Energies
  - EOS is determined with Mom. AND Density Dep. Pot. ?

Old but New (Continuing) Problem!
What is Collective Flow?

(Directed) Flow \((dP_x/dY)\)
- Stiffness (Low E)
- Time Scale (High E)

Elliptic Flow \((V_2)\)
- Thermalization & Pressure Gradient

Radial Flow \((\beta_T)\)
- Pressure History

\[
\epsilon \frac{DV}{Dt} = -\nabla P
\]
\[
\Rightarrow V = \int_{path} \frac{-\nabla P dt}{\epsilon}
\]

Until AGS

Above SPS

\(V_2 > 0:\) In Plane

\(V_2 < 0:\) Out of Plane

\(p_x, p_x, p_x, p_x\)
Side Flow at AGS Energies

- Relativistic BUU (RBUU) model: \( K \sim 300 \text{ MeV} \)
  
  (Sahu, Cassing, Mosel, AO, Nucl. Phys. A672 (2000), 376.)

- Boltzmann Equation Model (BEM): \( K=167\sim210 \text{ MeV} \)
  
Elliptic Flow

- What is Elliptic Flow? → Anisotropy in P space

- Hydrodynamical Picture
  - Sensitive to the Pressure Anisotropy in the Early Stage
  - Early Thermalization is Required for Large V2

Out-of-Plane Flow
\( (v_2 < 0) \)

\[
v_2 \equiv \left\langle \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2} \right\rangle = \langle \cos 2 \phi \rangle
\]

In-Plane Flow
\( (v_2 > 0) \)
Elliptic Flow at AGS

- Strong Squeezing Effects at low E (2-4 A GeV)
  - UrQMD: Hard EOS (S.Soff et al., nucl-th/9903061)
  - RBUU (Sahu-Cassing-Mosel-AO, 2000): K \sim 300 \text{ MeV}
  - BEM(Danielewicz2002): K = 167 \rightarrow 300 \text{ MeV}
Elliptic Flow from AGS to SPS

- JAM-MF with p dep. MF explains proton v2 at 1-158 A GeV
  - v2 is not very sensitive to K (incompressibility)
  - Data lies between MS(B) and MS(N)

![Graph showing Proton v2 for AGS to SPS Energies](image)
Dip of $V_2$ at 40 A GeV may be a signal of QCD phase transition at high baryon density. (Cassing et al.)

However, the data is too sensitive to the way of the analysis (reaction plane/two particle correlation).

- We have to wait for better data.
In addition to the ambiguities in in-medium cross sections, Res.-Res. cross sections, we have model dependence.

- **RBUU** *(e.g. Sahu, Cassing, Mosel, AO, Nucl. Phys. A672 (2000), 376.)*
  - In RMF, Strong cut-off for meson-N coupling in RMF
    → Smaller EOS dep.

- **Scalar potential interpretation in BUU**
  *Larionov, Cassing, Greiner, Mosel, PRC62, 064611('00), Danielewicz, NPA673, 375('00)*

  \[
  \varepsilon(p, \rho) = \sqrt{[m + U_s(p, \rho)]^2 + p^2} = \sqrt{m^2 + p^2 + U(p, \rho)}
  \]
  - Due to the Scalar potential nature, EOS dependence is smaller.

- **Scalar/Vector Combination** *Danielewicz, Lacey, Lynch, Science 298('02), 1592*

  \[
  \varepsilon(p, \rho) = m + \int_0^p dp' v^*(p', \rho) + \tilde{U}(\rho), \quad v^*(p, \rho) = \frac{p}{\sqrt{p^2 + [m^*(p, \rho)]^2}}.
  \]
  - Relatively Strong EOS dependence even at high energy

- **JAM-RQMD/S** *Isse, AO, Otuka, Sahu, Nara, PRC 72 (2005), 064908*
  - Similar to the Scalar model BUU